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Unstable solitons on noncommutative tori and D-branes*

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Abstract: We describe a class of exact solutions of super Yang-Mills theory on even-dimensional noncommutative tori. These solutions generalize the solitons on a noncommutative plane introduced in [7] that are conjectured to describe unstable D2p-D0 systems. We show that the spectrum of quadratic fluctuations around our solutions correctly reproduces the string spectrum of the D2p-D0 system in the Seiberg-Witten decoupling limit. In particular the fluctuations correctly reproduce the 0-0 string winding modes. For $p = 1$ and $p = 2$ we match the differences between the soliton energy and the energy of an appropriate SYM BPS state with the binding energies of D2-D0 and D4-D0 systems. We also give an example of a soliton that we conjecture describes branes of intermediate dimension on a torus such as a D2-D4 system on a four-torus.

Keywords: D-branes, Non-Commutative Geometry, Superstrings and Heterotic Strings.

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Contents

1. Introduction

The D0-D2p system in the presence of a constant background B-field was first studied in [1]. Later in [2] it was shown that in a certain decoupling limit the low energy effective theory on D2p-branes with a constant B-field can be described by means of a noncommutative super Yang-Mills (SYM) theory. The D0-D2p system in a B-field was further analyzed in [2, 3]. For a generic constant B-field such systems with point-like D0-branes are unstable. However for a certain locus of the B-field moduli we can have bound states with smeared D0-branes and in more restricted regions we can have bound states of point-like D0 branes sitting on top of the D2p-brane. Thus it was shown in [27] (see also [26, 28]) that the D0-D6 system, which is not supersymmetric in the absence of a B-field background, becomes supersymmetric for certain values of B.

In papers [4]–[9] a number of exact solitonic solutions in noncommutative SYM theory on a plane was constructed and their interpretation in terms of D-branes was given. In particular in [7] solitonic solutions describing unstable D0-D2 and D0-D4 systems were studied. It was shown that the solitons have a correct fluctuation spectrum and their energies match with the binding energies in the corresponding brane systems. These exact solutions describing non-BPS configurations are powerful tools in studying such systems, in particular tachyon condensation.

The aim of the present paper is to construct analogous solitonic solutions on tori in the presence of a large B-field background (understood in terms of the decoupling limit of [2]). To describe our approach to constructing such solitons let us first remind the reader the essential ingredients in the construction of SYM solitons on the plane as presented in the papers cited above.

First, our fields are functions on a noncommutative plane and they can be conveniently represented as operators acting on a Fock space. More precisely in complex coordinates on a two-plane (taken just for the purposes of illustration) one has the following representation of gauge fields (here we use the notations of [7])

$$A_z = iC - a^\dagger\,, \qquad A_{\bar z} = -i\bar C - a\,,$$

where a and $a^\dagger$ are annihilation and creation operators and C and $\bar C$ are essentially arbitrary operators (to specify a finite energy gauge field the fields A_z, $A_{\bar z}$ have to be gauge trivial at infinity, see [30] for a detailed discussion). In terms of the variables C, $\bar C$ the Yang-Mills equations of motion in the Coulomb gauge take the form

$$\partial_t^2 C = [C,[C,\bar C]]$$

plus the conjugated equation.

Next one can introduce the so called partial isometry operators S and $S^\dagger$ that satisfy

$$SS^\dagger = 1\,, \quad S^\dagger S = 1 - P_0\,,$$

where P_0 is a projector onto the Fock vacuum subspace. It is easy to see then that dressing transformations

$$C \mapsto (S^\dagger)^m a^\dagger S^m$$

generate solutions of the static equation $[C,[C,\bar C]] = 0$ and carry $m > 0$ units of magnetic flux.

Such a dressing transformation singles out a finite dimensional subspace in our Fock space. The projector on this subspace is $P_{m-1} = \sum_{i=0}^{m-1} |i\rangle\langle i|$. The curvature operator has nonvanishing components only for matrix elements between vectors from this subspace. If we consider fluctuations around our soliton then fluctuations with matrix elements between vectors from the subspace singled out by P_{m-1} describe strings attached to a D0 brane (that in this case seats at the origin). Fluctuations with matrix elements from the orthogonal complement describe D2-D2 strings and the off-diagonal ones describe D0-D2 strings.

As it can be easily deduced from this construction (see [17, sections 3.1, 3.2] for a pedagogical explanation) a generic module (representation) of the algebra of functions on a noncommutative plane denoted $\mathbb{R}_\theta$ has a form of a direct sum

$$(\mathcal{F})^m \oplus (\mathbb{R}_\theta)^n\,,$$

where $\mathcal{F}$ is a module that carries a unit of magnetic flux and is built on a Fock representation of the operator algebra $\mathbb{R}_\theta$ and $(\mathbb{R}_\theta)^n$ stands for n-copies of a free module, i.e. a copy of the algebra itself with an action by a (right or left) multiplication. This can be shown to hold for an arbitrary even-dimensional noncommutative plane with the numbers m and n being analogs of the instanton number and the rank of the gauge group, or respectively D0 and D2 brane charges.

The projector P_{m-1} can be viewed as a projector specifying a projective module, that is an analog of vector bundles in noncommutative geometry. For noncommutative tori

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such projectors specifying projective modules that carry nontrivial topological numbers are also known. In the two-dimensional case there is an explicit construction by Powers and Rieffel [15]. There are also other approaches to constructing projectors based on theta-functions [31, 32]. These constructions are certainly more complicated then the simple expression for the P_{m-1} above and we find them to be not very convenient from the computational point of view. On the other hand there is a different, more straightforward description of projective modules over noncommutative tori [14] when one simply specifies an action of the algebra generators on a Hilbert space. There is a simple class of projective modules that posses a constant curvature connection. These modules and connections on them can be used as building blocks to construct modules with various topological numbers and solitonic solutions on them. This will be our starting point in constructing solitons describing unstable D-brane systems on tori. Despite the fact that we will not be using explicit projector operators and partial isometries we will still be able effectively describe those solitons, compute their energy and fluctuations and analyze their moduli. We will see however (see the last section 5 of the paper) that this approach has some limitations and contrary to the unstable ones the BPS solutions can have a complicated description in the picture we use. And it would be desirable for the purposes of describing tachyon condensation to construct some analogs of partial isometry operators.

The paper is organized as follows. Our main example will be D0-D2 system on a two-torus. In section 2 we will describe projective modules describing a single D2, a single D0 and a D0-D2 system. In section 3 we introduce a solitonic solution on the D0-D2 module. We match the excess energy of the soliton above the energy of the SYM BPS state with the binding energy of the D0-D2 system, match the spectrum of fluctuations and discuss the solitons moduli space. In section 4 we introduce various generalizations of the system discussed in section 2. First we introduce a soliton on an arbitrary even-dimensional torus that generalizes the D0-D2 soliton. We compute the difference between the soliton energy and the SYM energy of the BPS state for the case of a four-torus and match it with the D0-D4 binding energy. We also show how one can construct solitons that we conjecture to describe a system with branes of intermediate dimension. We pick a D2-D4 system on a four-torus to illustrate this type of generalization. Finally, in that section we also discuss systems with multiples of branes of various dimension. In section 5 we discuss some open problems, in particular application of our results to tachyon condensation.

2. D2, D0 and D0-D2 modules

In this section we will introduce the basic mathematical constructions needed to describe D0 and D2 branes on noncommutative two-tori. We will be rather succinct and will only introduce the objects directly needed to describe the D0-D2 system. We refer the reader to the review papers [16], [17] for a more detailed exposition of the mathematical tools used here.

Consider a torus T^d with coordinates $x^i \sim x^i + 1$, $i = 1, \ldots, d$ and metric G_{ij}. Given an antisymmetric $d \times d$ tensor θ^{ij} we can define an algebra of functions on a noncommutative torus T_θ^d as an algebra with generators U_i satisfying the relations

$$U_i U_j = e^{2\pi i\theta^{ij}} U_j U_i \,. \tag{2.1}$$

It is convenient to define linear generators

$$U_{\mathbf{n}} = U_1^{n_1} U_2^{n_2} \cdot \cdots \cdot U_d^{n_d} e^{-\pi i \sum_{j<k} n_j n_k \theta^{jk}} \,, \tag{2.2}$$

where $\mathbf{n} = (n_1, \ldots, n_d)$, n_i are all integers and U_i^{-n}, $n_i > 0$ is understood as an n-th power of an inverse operator. A general function on T_θ^d can be represented as

$$f = \sum_{\mathbf{n}\in\mathbb{Z}^d} c(\mathbf{n}) U_{\mathbf{n}} \,,$$

where $c(\mathbf{n})$ are $\mathbb{C}$-valued coefficients.

Vector bundles over noncommutative tori are described by means of projective modules. Throughout the paper we work with right modules. To make the discussion more explicit in the remaining part of this section we will concentrate on two-dimensional tori. All of the constructions we will present below have a straightforward generalization for higher-dimensional tori to be discussed in section 4.

We can assume that a two-dimensional noncommutative torus is specified by a real number $\theta \geq 0$ so that $\theta^{ij} = \theta\epsilon^{ij}$. It is known that a projective module E over T_θ^2 is specified by two integers n, m such that $\dim(E) = n + m\theta > 0$ [13, 14]. These numbers specify a K-theory class of E so that n corresponds to the D2-brane charge and M to the D0-brane charge respectively. Below we describe two elementary modules, one carrying a unit of D2 brane charge and another one carrying a unit of D0 charge. Both modules were discussed informally in [2, section 6.2].

D2 module. This module to be denoted $E_{1,0}$ has topological numbers $n = 1$, $m = 0$ and is just a rank one free module. Its elements are elements of the T_θ^2 itself and the algebra acts by multiplication from the right. An arbitrary Yang-Mills field on $E_{1,0}$ can be written in the form $\partial_j + iu_j$ where ∂_j act according to $\partial_j(U_{\mathbf{n}}) = 2\pi i n_j U_{\mathbf{n}}$ and u_j are operators of left multiplication by elements $u_j \in T_\theta^2$ denoted by the same letters. We also assume that u_j are hermitian with respect to the involution $*$ that acts on the generators as $U_{\mathbf{n}}^* = U_{-\mathbf{n}}$.

D0 module. This module denoted $E_{0,1}$ has topological numbers $n = 0$ and $m = 1$. It can be explicitly defined in terms of generators U_1, U_2 acting on functions of one variable $\phi(x) \in L_2(\mathbb{R}^1)$ as follows

$$\phi(x)U_1 = \phi(x-\theta)\,, \qquad \phi(x)U_2 = e^{2\pi i x}\phi(x)\,. \tag{2.3}$$

The module $E_{0,1}$ has a constant curvature Yang-Mills field denoted $\tilde{\nabla}_j^0$ that acts as

$$\tilde{\nabla}_1^0\phi(x) = \frac{2\pi i}{\theta} x \cdot \phi(x)\,, \qquad \tilde{\nabla}_2^0\phi(x) = \frac{\partial\phi(x)}{\partial x}\,, \tag{2.4}$$

and the curvature is

$$F_{12} = [\tilde{\nabla}_1^0, \tilde{\nabla}_2^0] = -\frac{2\pi i}{\theta} \cdot \mathbf{1}\,, \tag{2.5}$$

where $\mathbf{1}$ stands for a unit operator on $E_{0,1}$.

An arbitrary connection can be represented in the form $\nabla_j = \tilde{\nabla}_j^0 + iz_j$ where z_j belongs to the algebra $\mathrm{End}_{T_\theta^2}(E_{0,1})$ of endomorphisms of the module $E_{0,1}$. It consists of all operators acting on $E_{0,1}$ that commute with the action of T_θ^2. The algebra $\mathrm{End}_{T_\theta^2}(E_{0,1})$ is itself an algebra of functions on the T-dual noncommutative torus $T_{1/\theta}$. Its generators Z_j act on $E_{0,1}$ as follows

$$Z_1\phi(x) = \phi(x+1)\,, \qquad Z_2\phi(x) = e^{2\pi i x/\theta}\phi(x)\,. \tag{2.6}$$

It is straightforward to check that the operators Z_i commute with the operators U_i given in (2.3) and satisfy the commutation relation

$$Z_1 Z_2 = e^{2\pi i/\theta} Z_2 Z_1\,.$$

Similarly to generators $U_{\mathbf{n}}$ (2.2) we can introduce a set of linear generators $Z_{\mathbf{n}}$ for $T_{1/\theta}^2$; one just needs to replace θ with $1/\theta$ and U's with Z's in (2.2). Then for an arbitrary connection $\tilde{\nabla}_j^0 + iz_j$ we can write

$$z_j = \sum_{\mathbf{n}\in\mathbb{Z}^2} a_j(\mathbf{n}) Z_{\mathbf{n}}\,,$$

where $a_j(\mathbf{n})$ are numbers. (As a point of mathematical rigor we should remark here that the functions $\phi(x)$ should be assumed to be smooth and decaying at infinity faster than any polynomial to ensure a well defined action of operators $\tilde{\nabla}_j^0$ and polynomials thereof.)

Fock space picture. Note that the operators (2.4) give a single copy of the irreducible representation of the Heisenberg algebra. The whole module $E_{0,1}$ can be described in terms of a Fock space that is very similar in spirit to the construction of D0-module on noncommutative $\mathbb{R}^2$. Introduce creation and annihilation operators as

$$\begin{aligned} a^\dagger &= \sqrt{\frac{\pi}{\theta}}x - \sqrt{\frac{\theta}{4\pi}}\frac{\partial}{\partial x} \\ a &= \sqrt{\frac{\pi}{\theta}}x + \sqrt{\frac{\theta}{4\pi}}\frac{\partial}{\partial x} \end{aligned} \tag{2.7}$$

then we have

$$\tilde{\nabla}_1^0 = i\sqrt{\frac{\pi}{\theta}}\left(a + a^\dagger\right)\,, \qquad \tilde{\nabla}_2^0 = \sqrt{\frac{\pi}{\theta}}\left(a - a^\dagger\right)\,. \tag{2.8}$$

The generators U_j of the torus and the generators Z_j of the dual torus can also be expressed in terms of a and $a^\dagger$ and for the vectors of $E_{0,1}$ instead of the Schroedinger representation $\phi(x)$ we can go to the occupation number representation.

D0-D2 module. In the next section we will work with a module $E = E_{0,1} \oplus E_{1,0}$ that is a direct sum of the two modules discussed above. Elements of this module can be represented as a pair $(\phi(x), a)$, $\phi(x) \in L_2(\mathbb{R}^1)$, $a \in T_\theta^2$ that we will write as a column vector

$$\begin{pmatrix} \phi(x) \\ a \end{pmatrix}.$$

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An arbitrary Yang-Mills field on such a module can be written in the following form

$$\nabla_j = \nabla_j^0 + i \begin{pmatrix} z_j & \psi_j \\ \psi_j^* & u_j \end{pmatrix} , \tag{2.9}$$

where

$$\nabla_j^0 = \begin{pmatrix} \tilde{\nabla}_j^0 & 0 \\ 0 & \partial_j \end{pmatrix}$$

is simply the direct sum of the two fiducial connections introduced above. As about the second term in (2.9) the diagonal blocks z_j and u_j were already described above, the off-diagonal blocks ψ_j and ψ_j^* should specify T_θ-linear mappings from the free module $E_{1,0}$ to $E_{0,1}$ and in the opposite direction respectively. It follows from T_θ-linearity that $\psi_j(a) = \psi_j(1 \cdot a) = \psi_j(1)a$ and hence to specify the mapping ψ_j we only need to know its value on the unit element $1 \in T_\theta^2$. The last one is a function from $L_2(\mathbb{R}^1)$ to be denoted $\psi_j(x)$. The construction of the conjugated mappings $\psi_j^* : E_{0,1} \to E_{1,0}$ is a bit more involved. Given a function $\phi(x)$ we need to map it into an element of the torus algebra in a way that commutes with the action of T_θ^2. This is done by using the T_θ^2-valued inner product $\langle , \rangle_{T_\theta^2}$:

$$\psi_j^*(\phi) = \langle \psi_j, \phi \rangle_{T_\theta^2} ,$$

where on the right hand side $\psi_j = \psi_j(x)$, $\phi = \phi(x)$ are both elements of $E_{0,1}$. The inner product at hand can be defined by the following explicit formula

$$\begin{aligned} \langle \psi, \phi \rangle_{T_\theta^2} &= \sum_{\mathbf{n} \in \mathbb{Z}^2} \langle \psi, \chi U_{-\mathbf{n}} \rangle_{\mathbb{C}} U_{\mathbf{n}} \\ &= \sum_{n_1, n_2} U_{\mathbf{n}} e^{-\pi i n_1 n_2 \theta} \int dx\, \bar{\psi}(x) \phi(x + n_1\theta) e^{-2\pi i x n_2} . \end{aligned} \tag{2.10}$$

It is easy to check that this inner product satisfies the desired T_θ^2-linearity property

$$\langle \psi, \phi a \rangle_{T_\theta^2} = \langle \psi, \phi \rangle_{T_\theta^2} \cdot a ,$$

where on the LHS a stands for the right action of $a \in T_\theta^2$ on $E_{0,1}$ specified in (2.3) and on the RHS we have a multiplication of two elements of T_θ^2.

According to the discussion in [2, section 6.3] the algebra T_θ^2 is the algebra of ground state 2-2 strings, $T_{1/\theta}^2$ is that of 0-0 strings and they both act naturally on the 0-2 strings, each at the corresponding end of the string. The 0-2 strings in their turn are described by elements of $E_{0,1} \cong L_2(\mathbb{R}^1)$ and the actions of 2-2 and 0-0 algebras are given by operators (2.3), (2.6). The fact that we have a left and a right action stems from the difference in outward orientations of the open string at $\sigma = 0$ and $\sigma = \pi$. On the Yang-Mills side all of this fits very well with the interpretation of the diagonal blocks in (2.9) as the ones describing excitations of 0-0 and 2-2 strings and the off-diagonal ones describing excitations of 0-2 strings.

In order to construct a SYM action functional on $E_{0,1} \oplus E_{1,0}$ we need to define a trace on the algebra of endomorphisms of this module, i.e. on operators having a 2×2 block

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form as in (2.9). It is defined as

$$\mathrm{Tr}\begin{pmatrix} z & \psi \\ \psi^* & u \end{pmatrix} = \mathrm{Tr}_{1/\theta} z + \mathrm{Tr}_\theta u \tag{2.11}$$

where $\mathrm{Tr}_{1/\theta}$ and Tr_θ are traces on the algebras $T^2_{1/\theta}$ and T^2_θ respectively defined as

$$\begin{aligned} \mathrm{Tr}_{1/\theta} \sum_{\mathbf{n}\in\mathbb{Z}^2} a(\mathbf{n}) Z_{\mathbf{n}} &= \theta a(0) \\ \mathrm{Tr}_\theta \sum_{\mathbf{n}\in\mathbb{Z}^2} c(\mathbf{n}) U_{\mathbf{n}} &= c(0) \end{aligned} \tag{2.12}$$

(see e.g. [16] for an explanation of the normalizations of the traces). When we multiply two operators having a block form as above and then take a trace the following identities [15, 14] are quite useful

$$\mathrm{Tr}_{1/\theta} \psi \circ \chi^* = \mathrm{Tr}_\theta \chi^* \circ \psi = \mathrm{Tr}_\theta \langle \chi, \psi \rangle_{T^2_\theta} = \langle \chi, \psi \rangle_{\mathbb{C}} \,. \tag{2.13}$$

3. Solitons on a two-torus and D0-D2 system

The soliton. We consider a SYM theory on $T^d_\theta \times \mathbb{R}^1$ where $\mathbb{R}^1$ corresponds to the time direction. The action functional in the Coulomb gauge has the form

$$\begin{aligned} S = & -\frac{\sqrt{G}}{4g^2_{YM}} \mathrm{Tr}\Big(-2\partial_t A_i G^{ij} \partial_t A_j + F_{ij} F_{kl} G^{ik} G^{jl} + 2[\nabla_i, X_I] G^{ij} [\nabla_j, X^I] + [X_I, X_J]^2 \Big) + \\ & + \text{fermionic term.} \end{aligned} \tag{3.1}$$

Here the indices i, j, k, l run from 1 to d, the indices I, J run from $d+1$ to 9, $\nabla_i = \nabla^0_i + iA_i$ stand for the Yang-Mills fields and X_I for the scalar fields, G^{ij} is the torus metric and g_{YM} is the Yang-Mills coupling constant.

To describe a D0-D2 system on T^2_θ we consider the action functional (3.1) for $d = 2$ with the fields acting on a module $E_{1,1} \cong E_{0,1} \oplus E_{1,0}$ described in the previous section. There exists a $\frac{1}{2}$-BPS solution on this module describing a D0 brane dissolved inside a D2 brane wrapped on T^2_θ. Our prime interest here is however in an unstable soliton describing a point-like D0-brane sitting on top of D2 brane (or more generally being displaced away in the transverse to the D2-brane direction by some small distance). This soliton is a torus analog of the solitonic solution of [7] on a noncommutative two-plane. Below we will refer to the unstable configuration as an undissolved D0 on D2 system. The BPS configuration describes a bound state that should be the end point of tachyon condensation in the unstable system. We will have a further discussion of the BPS state later when computing the soliton binding energy.

We propose to use the following static solution of SYM equations to describe the undissolved D0 on D2 system on T^2_θ

$$\nabla_j = \begin{pmatrix} \tilde{\nabla}^0_j + ic_j \cdot \mathbf{1} & 0 \\ 0 & \partial_j \end{pmatrix}, \quad X^I = \begin{pmatrix} x^I \cdot \mathbf{1} & 0 \\ 0 & 0 \end{pmatrix}. \tag{3.2}$$

Here the constants c_j describe the position of D0 brane on a torus and x^I's are its transverse coordinates (see a more detailed discussion later).

The curvature of this solution has a block form

$$F_{12} = \begin{pmatrix} -\frac{2\pi i}{\theta}\mathbf{1} & 0 \\ 0 & 0 \end{pmatrix}$$

and the energy is

$$E_{\rm sol} = \frac{(2\pi)^2}{4g_{YM}^2}\sqrt{\det(G\theta)}\left(\frac{1}{\theta}\right)_{ij} G^{ik}G^{jl}\left(\frac{1}{\theta}\right)_{lk}. \tag{3.3}$$

In two dimensions this expression reduces to

$$\frac{(2\pi)^2}{2g_{YM}^2\sqrt{\det G}\theta}. \tag{3.4}$$

Soliton energy vs. D0-D2 binding energy. The excess of the soliton energy (3.4) above the Yang-Mills energy of the bound state should be compared to the binding energy of the D0-D2 system in a constant B-field. Note that on the plane the process of tachyon condensation goes through a spreading of the D0-brane flux over the infinite volume of the plane. The Yang-Mills energy of the limiting bound state vanishes and can be neglected in the computation. This is not the case for the torus. The BPS solution (the explicit form of which will be discussed in the last section) has a constant curvature that in the block decomposition form can be written as

$$F_{12} = \begin{pmatrix} -\frac{2\pi i}{\theta+1}\mathbf{1} & 0 \\ 0 & -\frac{2\pi i}{\theta+1}\mathbf{1} \end{pmatrix}.$$

Its energy is

$$E_{BPS}^{SYM} = \frac{(2\pi)^2}{2g_{YM}^2\sqrt{\det G}(\theta+1)}$$

and the soliton excess energy is

$$\Delta E = E_{\rm sol} - E_{BPS}^{SYM} = \frac{(2\pi)^2}{2g_{YM}^2\sqrt{\det G}\theta(\theta+1)}. \tag{3.5}$$

To compute the D0-D2 binding energy in the presence of B-field we can use the Born-Infeld energy. In the Seiberg-Witten decoupling limit

$$\alpha' \sim \epsilon^{1/2} \to 0\,, \qquad g_{ij} \sim \epsilon \to 0 \tag{3.6}$$

the energy of the D2-brane has an expansion

$$E_{D2} = T_2\sqrt{\det g + \det(2\pi\alpha' B)} = T_2 2\pi\alpha' B + \left(\frac{T_2}{2}\right)\frac{\det g}{2\pi\alpha' B} + O(\epsilon)$$

where $T_2 = (g_s(2\pi)^2(\alpha')^{3/2})^{-1}$ is the 2-brane tension. The energy of the bound state can be evaluated from the above Born-Infeld by adding the corresponding constant field strength as

$$E_{D2/D0} = T_2\sqrt{\det g + \det(2\pi\alpha'(B+2\pi))} = T_2 2\pi\alpha' B + \left(\frac{T_2}{2}\right)\frac{\det g}{2\pi\alpha'(B+2\pi)} + O(\epsilon)\,.$$

Thus the D0-D2 binding energy is

$$E_{\rm bind} = T_0 + E_{D2} - E_{D2/D0} = \frac{T_2\, {\rm det} g}{2\alpha' B(B+2\pi)}\,. \tag{3.7}$$

To compare this expression with the soliton excess energy we rewrite (3.5) in terms of closed string parameters B and g. It follows from our periodicity conventions $x^i \sim x^i + 1$ that $2\pi/\theta = B$. In the Seiberg-Witten limit (3.6) on a D2p-brane we have in the leading order [2]

$$\begin{aligned} G_{ij} &= -(2\pi\alpha')^2 (Bg^{-1}B)_{ij}\,, \\ \frac{1}{g^2_{YM}} &= \frac{\sqrt{{\rm det} g}}{|{\rm Pf}B| g_s (\alpha')^{2p-3/2}(2\pi)^{3p-2}} = \frac{\sqrt{{\rm det}g}T_{2p}}{|{\rm Pf}B|(2\pi)^{p-2}(\alpha')^{p-2}}\,. \end{aligned} \tag{3.8}$$

Taking $p = 1$ and substituting these expressions in (3.5) we find an exact agreement with (3.7).

This result supports our proposal that the solution (3.2) describes the undissolved D0 on D2 system on a torus with a uniform magnetic field turned on. A further support comes from matching the spectrum of quadratic fluctuations around (3.2) with that of the D0-D2 excitations in the decoupling limit (3.6) taken on a torus.

Fluctuations. Let us represent fluctuations around (3.2) in the form

$$\delta A_j = \begin{pmatrix} \delta z_j & \delta\psi_j \\ \delta\psi^*_j & \delta u_j \end{pmatrix}, \qquad \delta X^I = \begin{pmatrix} \delta a^I & \delta\chi^I \\ \delta\chi^{*I} & \delta b^I \end{pmatrix}.$$

The Gauss constraint

$$[\nabla_i, \partial_t A^i] + [X_I, \partial_t X^I] = 0$$

being expanded to the first order in fluctuations gives

$$\begin{aligned} &\partial_t([\tilde\nabla^0_i, \delta z^i]) = 0\,, \qquad \partial_t([\partial_i, \delta u^i]) = 0\,, \\ &\partial_t(\tilde\nabla^0_i \delta\psi^i + ic_i\delta\psi^i + x_I\delta\chi^I) = 0\,. \end{aligned} \tag{3.9}$$

The calculation of the second derivative matrix is similar to the analogous computation on the plane. There is only one point in the computation that we would like to highlight. When encountering expressions of the form ${\rm Tr}_{1/\theta}\delta\psi_i \circ \delta\psi^*_j$ or ${\rm Tr}_\theta \delta\chi^*_i \circ \delta\chi_j$ we use the identity (2.13) that gives us $\langle \delta\psi_j, \delta\psi_i\rangle_{\mathbb{C}}$ and $\langle\delta\chi_i, \delta\chi_j\rangle_{\mathbb{C}}$ respectively. With this in mind it is straightforward to obtain

$$-\frac{1}{2}\delta^2 S = \frac{\sqrt{G}}{g^2_{YM}}(K_{2-2} + K_{0-0} + K_{0-2})\,,$$

where

$$K_{2-2} = {\rm Tr}_\theta\Big(\frac{1}{2}(\partial_t \delta u_i)^2 + \frac{1}{2}(\partial_t\delta b^I)^2 + \frac{1}{4}([\partial_i, \delta u_j] - [\partial_i, \delta u_j])^2 + \frac{1}{2}[\partial_i, \delta b^I]^2\Big) \tag{3.10}$$

$$K_{0-0} = {\rm Tr}_{1/\theta}\Big(\frac{1}{2}(\partial_t\delta z_i)^2 + \frac{1}{2}(\partial_t\delta a^I)^2 + \frac{1}{4}([\tilde\nabla^0_i, \delta z_j] - [\tilde\nabla^0_j, \delta z_i])^2 + \frac{1}{2}[\tilde\nabla^0_i, \delta a^I]^2\Big) \tag{3.11}$$

$$\begin{aligned} K_{0-2} = {} & \|\partial_t\delta\psi^i\|^2_{\mathbb{C}} + \|\partial_t\delta\chi^I\|^2_{\mathbb{C}} + \frac{1}{2}\|\tilde\nabla^0_i\delta\psi_j - \tilde\nabla_j\delta\psi_i + i(c_i\delta\psi_j - c_j\delta\psi_i)\|^2_{\mathbb{C}} - \\ & -({\rm det}G)^{-1}\frac{2\pi i}{\theta}(\langle\delta\psi_2, \delta\psi_1\rangle_{\mathbb{C}} - \langle\delta\psi_1, \delta\psi_2\rangle_{\mathbb{C}}) + \|\tilde\nabla^0_i\delta\chi^I + ic_i\delta\chi^I - ix^I\delta\psi_i\|^2_{\mathbb{C}}\,, \end{aligned} \tag{3.12}$$

where all of the summations in indices i, j are performed with the metric tensor G^{ij}.

It is not hard to see from the expression for K_{0-2} that despite a more complicated structure (2.10) of the off-diagonal terms, in the matrix of quadratic fluctuations their contribution is exactly the same as on the plane [7, 10]. This happens due to identities (2.13). This of course goes along well with our physical interpretation because on the torus the 0-2 strings have the same spectrum as on the plane. Our analysis of (3.12) is essentially the same as in [7]. In order to have a complete discussion and bearing in mind the difference in notation that might obscure the correspondence with [7] we would like to sketch here the analysis of $K_{0,2}$. For simplicity let us first consider the case $x^I = 0$ when D0 seats on top of D2. Let us also assume here $G_{ij} = G\delta_{ij}$ (we will restore a generic constant metric later in the analysis of diagonal fluctuations where it will be truly important). Recall that in the Fock space picture of the module $E_{0,1}$ the operators $\tilde{\nabla}^0_j$ are expressed via creation and annihilation operators according to (2.8). It is easy to see that the spectrum of fluctuations does not depend on the moduli c_j as one can absorb them into redefined creation and annihilation operators related to the ones introduced in (2.7) by a suitable unitary transformation on $L_2(\mathbb{R}^1)$. Changing the variables

$$\delta\psi_1 = i(\delta\psi_- - \delta\psi_+)\frac{1}{\sqrt{2}}\,, \qquad \delta\psi_2 = (\delta\psi_- + \delta\psi_+)\frac{1}{\sqrt{2}}$$

and using (2.8) we obtain for the potential term in (3.12)

$$\frac{2\pi}{\theta G^2}(\|a\delta\psi_- + a^\dagger\delta\psi_+\|^2 + \|\delta\psi_+\|^2 - \|\delta\psi_-\|^2 + G\langle\delta\chi^I, (1 + 2a^\dagger a)\delta\chi^I\rangle_{\mathbb{C}})\,.$$

Expanding

$$\delta\psi_\pm = \sum_{n\geq 0} t_n^\pm|n\rangle\,, \quad \delta\chi^I = \sum_{n\geq 0} s_n^I|n\rangle$$

and substituting into the above expression we obtain

$$\begin{aligned} K_{0-2} = &\sum_{n=0}^{\infty}\frac{1}{2G}|\partial_t t_n^+|^2 + \frac{1}{2G}|\partial_t t_n^-|^2 + \frac{1}{2}|\partial_t s_n^I|^2 + \\ &+\frac{2\pi}{\theta G^2}\Big(\sum_{n=0}^{\infty}|\sqrt{n}\,t_{n-1}^+ + \sqrt{n+1}\,t_{n+1}^-|^2 + |t_n^+|^2 - |t_n^-|^2 + G(2n+1)|s_n^I|^2\Big)\,. \end{aligned}$$

It follows from this expression that the bosonic spectrum of 0-2 strings contains a tachyon mode t_0^- with mass squared $m^2_{\rm tach} = -2\pi/(G\theta) = -g(2\pi\alpha')^{-2}B^{-1}$, a tower of positive mass modes

$$\xi_n = \frac{1}{\sqrt{2n+1}}\left(\sqrt{n+1}\,t_{n-1}^+ + \sqrt{n}\,t_{n+1}^-\right)$$

with $m^2 = (2n+1)g(2\pi\alpha')^{-2}B^{-1}$, the same tower for each transverse scalar χ^I and an infinite number of zero modes

$$\eta_n = \frac{1}{\sqrt{2n+1}}\left(\sqrt{n}\,t_{n-1}^+ - \sqrt{n+1}\,t_{n+1}^-\right)$$

that are nondynamical due to the Gauss law constraint (3.9). The analysis of 0-2 strings in the Seiberg-Witten decoupling limit is exactly the same on the torus as on the plane, thus as it was shown in [7] we have a match of the fluctuation spectra in this sector. It is easy to see from (3.12) that switching on the moduli x^I shifts all squared masses up by $x^I x_I$. This matches with the shift in the spectra of D0-D2 system that occurs when we displace D0 in a direction transverse to the torus by the vector $2\pi\alpha' x^I$.

Let us discuss next the spectrum of fluctuations entering the K_{0-0} term. Expanding

$$\delta z_j = \sum_{\mathbf{n}\in\mathbb{Z}^2} q_j(\mathbf{n}) Z_{\mathbf{n}} \,, \qquad \delta a^I = \sum_{\mathbf{n}\in\mathbb{Z}^2} r^I(\mathbf{n}) Z_{\mathbf{n}}$$

imposing the Gauss law constraint (3.9) and using the formula

$$[\tilde{\nabla}^0_j, Z_{\mathbf{n}}] = \frac{2\pi i}{\theta} n_j Z_{\mathbf{n}} \,, \tag{3.13}$$

that follows from (2.6), (2.4), we obtain that the fluctuations are described in terms of a set of 8 towers of oscillators with frequencies

$$\omega^2(\mathbf{n}) = \left(\frac{2\pi}{\theta}\right)^2 n_i G^{ij} n_j \tag{3.14}$$

and polarization transverse to $\mathbf{n}$. On the other hand, for a D0-brane on a torus with closed string metric g_{ij} and radii $r = 1/2\pi$ the winding modes w^i contribute to the spectrum as

$$m^2 = \left(\frac{1}{2\pi\alpha'}\right)^2 w^i g_{ij} w^j + \frac{1}{\alpha'}(N - a) \,.$$

In the Seiberg-Witten limit the surviving states in the bosonic sector are $e_m \psi^m_{-1/2}|0, w_1, w_2\rangle_{NS}$ where e_m is the polarization vector for the collective coordinates of D0-brane. We have therefore 8 transverse oscillators with

$$m^2 = -w^i (BGB)_{ij} w^j = \frac{1}{\theta^2} w^i \epsilon_{ij} G^{jk} \epsilon_{kl} w^l$$

that matches with (3.14) upon identification $n_i = \epsilon_{ij} w^j$.

The term K_{2-2} similarly gives us a tower of K-K modes of noncommutative Yang-Mills theory with 7 adjoint scalars on a torus that is precisely what we expect for excitations of a wrapped D2-brane. Thus we see that the spectra in all sectors match.

Moduli. Now we would like to discuss the moduli c_i. It is clear that these moduli should come from the positions of D0-branes on the original shrinking torus. The analogous moduli and their interpretation for the case of noncommutative plane were discussed in [7, 9, 34, 10, 33] and more recently in [11]. On the torus we need to check the periodicities of c_j and a metric on them. We first note that due to gauge transformations

$$Z^\dagger_{\mathbf{n}} (\tilde{\nabla}^0_j + i c_j \mathbf{1}) Z_{\mathbf{n}} = \tilde{\nabla}^0_j + i\left(c_j + \frac{2\pi}{\theta} n_j\right)\mathbf{1}$$

the moduli c_i are periodic with periods $2\pi/\theta$. A natural metric on moduli c_i is G^{ij}. Probably the most convincing way to check the last statement is to consider a module representing two D0 branes on top of a D2-brane (more on these generalized solutions in the next section 4). It is given by a direct sum of two copies of D0-module $E_{0,1}$ and a single copy of a D2-module $E_{1,0}$. Then the appropriate soliton has a form (3.2) with an upper block replaced by a two-by-two block

$$\begin{pmatrix} \tilde{\nabla}_j^0 + ic_j^{(1)}\mathbf{1} & 0 \\ 0 & \tilde{\nabla}_j^0 + ic_j^{(2)}\mathbf{1} \end{pmatrix}.$$

It can be checked then that the off-diagonal fluctuations in the above block are massive with masses

$$m^2 = \left(c_i^{(1)} - c_i^{(2)}\right) G^{ij} \left(c_j^{(1)} - c_j^{(2)}\right).$$

If we introduce now rescaled coordinates $\tilde{c}_j = \frac{\theta}{2\pi} c_j$ that have periodicities 1, then the metric on these coordinates is $\frac{2\pi}{\theta} G^{ij}$. On the other hand for the strings stretched between two D0 branes on a torus located at points x_1^i and x_2^i we have the mass spectrum of the form

$$m^2 = \left(\frac{2\pi}{\alpha'}\right)^2 (x_1^i - x_2^i) g_{ij} (x_1^j - x_2^j) + \frac{1}{\alpha'}(N - a)$$

that in the Seiberg-Witten limit gives the metric $-\frac{2\pi}{\theta}\epsilon_{ik} G^{kl} \epsilon_{lj}$. Thus we can identify

$$c_i^a = \frac{2\pi}{\theta} \epsilon_{ij} x_a^j \,. \tag{3.15}$$

Note that we could also switch on Wilson lines on the D2-brane by taking $\partial_j + is_j\mathbf{1}$ instead of ∂_j in (3.2). In that case a gauge transformation

$$U_{\mathbf{n}}^\dagger(\partial_j + is_j\mathbf{1})U_{\mathbf{n}} = \partial_j + i(s_j + 2\pi n_j)\mathbf{1}$$

implies that the moduli s_j have periodicities 2π. A natural metric on them is G^{ij}. It can be checked that in the Seiberg-Witten limit

$$\left(\frac{2\pi}{\alpha'}\right)^2 g'_{ij} \to -\frac{1}{(\alpha')^2}\left(\frac{1}{B} g \frac{1}{B}\right)^{ij} = (2\pi)^2 G^{ij} \,,$$

where

$$g'_{ij} = \left(\frac{\alpha'}{g - 2\pi\alpha' B}\, g\, \frac{\alpha'}{g + 2\pi\alpha' B}\right)_{ij}$$

is a metric obtained from g_{ij}, B_{ij} by a T-duality in all sides of the torus. Thus the moduli s_j specify a point on the T-dual torus as it should be in the usual interpretation of Wilson lines.

Finally we would like to remark that the most direct way to see the localization of D0-brane and to check the metric on its positions moduli would be through the use of Seiberg-Witten transformation. Recently the Seiberg-Witten transform of noncommutative solitons and instantons on a plane was studied in papers [24, 25].

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4. Generalizations

In this section we will consider both generalizations for higher dimensional tori and for systems with multiple numbers of branes.

We start with a solution describing an undissolved D0-brane seating on top of D2p brane wrapped on T^{2p}. For simplicity we will assume that our noncommutativity matrix is brought to the canonical form

$$(\theta^{ij}) = \begin{pmatrix} \theta_1\epsilon & 0 & \dots & 0 \\ 0 & \theta_2\epsilon & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \theta_p\epsilon \end{pmatrix}, \tag{4.1}$$

where

$$\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

A D2p-module is again just a rank one free module over T^{2p}_θ. A D0-module is constructed by defining the action of torus generators on functions of p-variables $x^1, \dots, x^p$ as

$$\phi(x)U_{2j+1} = \phi(x^1, \dots, x^j - \theta_j, \dots x^p)\,, \qquad \phi(x)U_{2j} = e^{2\pi i\theta_j x^j}\phi(x)\,,\ \ j = 1, \dots, p\,. \tag{4.2}$$

This module has a constant curvature connection

$$\tilde{\nabla}^0_{2j+1}\phi(x) = \frac{2\pi i}{\theta_j}x^j \cdot \phi(x)\,, \qquad \tilde{\nabla}^0_{2j}\phi(x) = \partial_j\phi(x) \tag{4.3}$$

with the curvature tensor

$$[\tilde{\nabla}^0_j, \tilde{\nabla}^0_k] = 2\pi i(\theta^{-1})_{jk}\mathbf{1}\,.$$

By computing the Chern character and then using Elliott's formula [12] (see e.g. [16] for an explanation of these tools in noncommutative geometry) one can show that this module has a D0 number equal to 1 and no other charges [21]. Next we take a module that is a direct sum of the D0-module (4.2) and a D2p-module and consider a SYM solution that upon using the connection (4.3) for $\tilde{\nabla}^0_j$ has exactly the same form as (3.2). Its energy is given by (3.3) where G and θ are now understood as $2p \times 2p$ matrices.

Soliton energy and D0-D4 binding energy. Next we would like to compute the soliton excess energy above the BPS state for $p = 2$ and compare it to the binding energy of the D0-D4 system. The BPS solution is a noncommutative instanton. It preserves $\frac{1}{4}$ of supersymmetries. Its energy can be computed from the noncommutative SYM supersymmetry algebra and reads [22]

$$E^{SYM}_{BPS} = \frac{\sqrt{\det G}\pi^2}{g^2_{YM}(1 + \mathrm{Pf}\theta)}|\mathrm{tr}(\tilde{\theta}G^{-1}\tilde{\theta}G^{-1})| + \frac{(2\pi)^2}{g^2_{YM}(1 + \mathrm{Pf}\theta)}\,, \tag{4.4}$$

where $\tilde{\theta}_{ij} = \frac{1}{2}\epsilon_{ijkl}\theta^{kl}$ and $\mathrm{Pf}\theta$ is assumed to be positive. Let us explain here where this choice comes from. For a general module E over T^4_θ that carries n D4, m D0 and no other charges the noncommutative dimension is $\dim(E) = n + m\mathrm{Pf}\theta$ and it has to be positive, otherwise this charges specify just a virtual module (a representative of the K_0 group). When $n = 1$ and $m = 1$ we can either have $\mathrm{Pf}\theta > 0$ or $-1 < \mathrm{Pf}\theta < 0$. In the last case however the module cannot be represented as a direct sum of a D4 and a D0 module, but rather as a difference (in the sense of K-theory). Thus to have simultaneously both the undissolved and the dissolved BPS solutions on our module we should take $\mathrm{Pf}\theta > 0$. This choice also follows from the analysis of the binding energy made below. The system with $-1 < \mathrm{Pf}\theta < 0$ behaves rather like a $\overline{\mathrm{D0}}$-D4 system (see more on this at the end of this section). The qualitatively different behavior for different signs of $\mathrm{Pf}B$ in the D0-D4 system is discussed in [2, 29]. Note that in this paper we have the opposite sign convention than in those two papers (this comes from the use of Elliott's formula (4.11)).

The soliton excess energy above the BPS state can be computed from (3.3), (4.4) to be

$$\Delta E = \frac{\sqrt{\det G}\pi^2}{g_{YM}^2 \mathrm{Pf}\theta(1+\mathrm{Pf}\theta)}|\mathrm{tr}(\tilde{\theta}G^{-1}\tilde{\theta}G^{-1})| - \frac{(2\pi)^2}{g_{YM}^2(1+\mathrm{Pf}\theta)}\,, \tag{4.5}$$

where we used the identity

$$\mathrm{tr}\,\frac{1}{\theta}G^{-1}\frac{1}{\theta}G^{-1} = \frac{\mathrm{tr}\,\tilde{\theta}G^{-1}\tilde{\theta}G^{-1}}{(\mathrm{Pf}\theta)^2}\,.$$

The mass squared for a $\frac{1}{4}$ BPS state with N_4 D4-branes and N_0 D0-branes in the presence of B-field is given by the following expression taken from [23] (see their formula (3.42))

$$\begin{aligned} M^2 \, = \, & T_4^2\Big[N_4^2\det g + (2\pi\alpha')^4(N_4\mathrm{Pf}B + (2\pi)^2N_0)^2 + \\ & + \frac{1}{2}N_4^2(2\pi\alpha')^2|\mathrm{tr}\,\tilde{B}g\tilde{B}g| + 2(2\pi\alpha')^2(2\pi)^2|N_0N_4|\sqrt{\det g}\Big]\,. \end{aligned} \tag{4.6}$$

For $N_0 = 1$, $N_4 = 1$ we have in the limit (3.6)

$$E_{D4/D0} = T_4(2\pi\alpha')^2(\mathrm{Pf}B + (2\pi)^2) + \frac{T_4}{(\mathrm{Pf}B + (2\pi)^2)}\left[\frac{1}{4}|\mathrm{tr}\,\tilde{B}g\tilde{B}g| + (2\pi)^2\sqrt{\det g}\right] + O(\epsilon)\,. \tag{4.7}$$

On the other hand for the energy of a single D4-brane in a B-field we have from (4.6 (or equivalently from Born-Infeld action)

$$E_{D4} = T_4(2\pi\alpha')^2\mathrm{Pf}B + \frac{T_4|\mathrm{tr}\,\tilde{B}g\tilde{B}g|}{4\mathrm{Pf}B} + O(\epsilon)\,. \tag{4.8}$$

Taking the difference we obtain for the binding energy in the leading order

$$E_{\mathrm{bind}} = T_0 + E_{D4} - E_{D4/D0} = \frac{(2\pi)^2T_4}{(\mathrm{Pf}B + (2\pi)^2)}\left[\frac{|\mathrm{tr}\,\tilde{B}g\tilde{B}g|}{4\mathrm{Pf}B} - \sqrt{\det g}\right]. \tag{4.9}$$

Upon expressing (4.5) in terms of closed string parameters (3.8) we find again an exact agreement with the binding energy (4.9). When checking this we found the following identity useful

$$\mathrm{tr}\,\tilde{B}g\tilde{B}g = \det g\,\mathrm{tr}\,Bg^{-1}Bg^{-1}\,.$$

The analysis of fluctuations around the $2p$-dimensional soliton is parallel to the analysis for the two-dimensional case done in the previous section. In fact for the 0-0 and 4-4 strings the results for $2p = 2$ can be generalized in a straightforward manner. As for the 0-4 strings it is simple to show using (2.13) that the quadratic fluctuation matrix in the end reduces to the one on the plane. We can use then the results of paper [10] in which the agreement of the soliton fluctuation spectrum and that of the 0-2p string excitations was checked for $p = 1, 2, 3, 4$.

Furthermore, in analogy with formulae (2.6) and (3.13) we can introduce a basis $Z_{\mathbf{n}}$ for the endomorphisms (generators of the 0-0 strings algebra) such that

$$[\nabla_j, Z_{\mathbf{n}}] = 2\pi i \theta^{-1}_{jk} n^k Z_{\mathbf{n}} \,.$$

Then it is not hard to see that our discussion of the moduli generalizes to the case at hand and yields an identification

$$c_j = 2\pi \theta^{-1}_{jk} x^k$$

analogous to (3.15). A similar statement is also true for the Wilson lines moduli on D2p.

Branes of intermediate dimension. So far we were discussing systems that contained branes of extremal dimensions. And these type of systems are in some sense direct descendants of the similar systems on a plane (sphere). It would be more interesting to see solitons describing undissolved branes of intermediate dimension. Here we would like to propose a module and a soliton that we conjecture describes a two-brane on top of a four-brane on T^4_θ. It will be clear from this particular example how one can generalize this solution to other systems with intermediate branes.

First note that with the choice of θ^{ij} in the form (4.1) the algebra of functions on a four-dimensional noncommutative torus T^4_θ is actually a tensor product $T^2_{\theta_1} \otimes T^2_{\theta_2}$ of two mutually commuting algebras of functions on two-dimensional tori. Thus we can consider modules that are tensor products $F \otimes G$. Here F is a module over $T^2_{\theta_1}$ and G is a module over $T^2_{\theta_2}$. Now take $F = E_{0,1}$, $G = E_{1,0}$ where $E_{0,1}$ and $E_{1,0}$ are modules introduced in section 2. The Chern character for $E = F \otimes G$ is a product of the Chern characters of the factors: $\mathrm{ch}(E) = \mathrm{ch}(F)\mathrm{ch}(G) = 1 \cdot (1 + \alpha^2 \wedge \alpha^3) = 1 + \alpha^2\alpha^3$ where α^i stand for the noncommutative analogs of the integer cohomology generators. Thus we see that E carries charges corresponding to a D2-brane wrapped on the (1,2)-cycle. Furthermore this module has a connection

$$\tilde{\nabla}_i = \tilde{\nabla}^0_i \otimes \mathbf{1} \,, \quad \text{for} \;\; i = 1, 2 \qquad \tilde{\nabla}_i = \mathbf{1} \otimes \partial_i \,, \quad \text{for} \;\; i = 3, 4$$

where $\tilde{\nabla}^0_i$ is given by (2.4). This connection has a curvature tensor whose only nonvanishing components are $F_{12} = -F_{21} = -\frac{2\pi i}{\theta_1} \cdot 1$. To describe a D2-D4 system let us take a direct sum $E = E_1 \oplus E_2$ of a module we have just described, to be denoted E_1, and a rank one free module $E_2 = (T^4_\theta)^1$. Let us take a Yang-Mills field ∇_i to be a direct sum of $\tilde{\nabla}_i$ described above and a standard connection ∂_i on E_2. Then it is straightforward to see that we have a solution to the Yang-Mills equations of motion (for simplicity we set here to zero the analogs of moduli c_i and x^I). We also know that this module carries the correct D2 and D4 charges. Furthermore it is not hard to see using the results we already obtained that we have the correct spectrum of fluctuations. Let us sketch how it works. Assume for simplicity that the metric G^{ij} also has a block-diagonal form with diagonal blocks giving a metric on each T^2. Fluctuations around the soliton have schematically the following form

$$\delta A_j = \begin{pmatrix} \delta u_j \otimes \delta z_j & \delta u'_j \otimes \delta\psi_j \\ (\delta u'_j)^* \otimes \delta\psi^*_j & \delta u''_j \end{pmatrix},$$

where $\delta z_j : F \to F$ are endomorphisms of F (if we forget about the second torus these are elements of the 0-0 strings algebra on $T^2_{\theta_2}$), $\delta\psi_j$ are elements of F itself, $\delta u_j, \delta u'_j \in T^2_{\theta_2}$, $\delta u''_j \in T^4_\theta$. The string spectrum in the D2-D4 system is also built as a tensor product of spectra for individual directions x^1, x^2, x^3, x^4. It is clear then that we should identify the fluctuations δz_j and δu_j with those of D-D and N-N directions in the 2-2 string excitations, $\delta\psi_j$ and $\delta u'_j$ with D-N and N-N directions for 2-4 strings, and $\delta u''_j$ with excitations of 4-4 strings. Indeed, take for example the 2-2 sector. We showed already that the modes δz_j on $T^2_{\theta_2}$ give the correct spectrum of D0-D0 winding modes. And this is precisely what we should have for the two D-D directions of the 2-2 strings as they can wind around the second torus. It is easy to see that the other sectors match as well.

Multiple branes. Once we constructed individual D2p-brane modules it is simple to construct modules describing multiple brane systems. To that end one simply takes direct sums of modules and elementary solutions on each one of them.

For example if we would like to describe a system of m D0-branes sitting on top of a stack of n D2 branes wrapped on a two-torus we take a direct sum of m copies of D0 module and n copies of the D2 module. The four blocks in (3.2) get enlarged to $m \times m$, $m \times n$ and $n \times n$ matrices as

$$\nabla_j = \begin{pmatrix} \tilde{\nabla}_j \otimes \mathbf{1}_{m\times m} + ic_j & 0 \\ 0 & \partial_j \otimes \mathbf{1}_{n\times n} \end{pmatrix}, \qquad X^I = \begin{pmatrix} x^I & 0 \\ 0 & 0 \end{pmatrix}, \tag{4.10}$$

where c_j and x^I are now $m \times m$ diagonal matrices whose eigenvalues describe positions of m D0 branes. The analysis of fluctuations for this and analogous systems is a direct generalization of the one we described in section 2. Fluctuations in the upper and lower diagonal blocks in (4.10 are those of noncommutative Yang-Mills theories with gauge groups $U(m)$ and $U(n)$ respectively and with gauge fields taking values in the appropriate algebras: in the T_θ and in the T-dual torus $T_{1/\theta}$. They describe the corresponding sectors of various D0-D0 and D2-D2 strings. The off diagonal terms correspond to the $n \times m$ sectors of D0-D2 strings each one described by a vector $\delta\psi_{ij} \in L_2(\mathbb{R}^1)$.

A general projective module E over T^{2p}_θ is described by a collection of charges that can be written out as an inhomogeneous exterior form with integer coefficients

$$\mu(E) = n + \frac{1}{2} m_{ij}\alpha^i \wedge \alpha^j + \frac{1}{4!} m^{ijkl}\alpha^i \wedge \alpha^j \wedge \alpha^k \wedge \alpha^l + \cdots + m\alpha^1 \wedge \cdots \wedge \alpha^{2p}.$$

The noncommutative dimension $\dim(E)$ of a module with such charges can be evaluated via Elliott's formula [12]. One should apply to $\mu(E)$ an operator

$$\exp\left(-\frac{\partial}{\partial\alpha_j}\theta^{jk}\frac{\partial}{\partial\alpha_k}\right) \tag{4.11}$$

and take the zero grading component. The condition for $\mu(E)$ to give topological numbers of a projective module is $\dim(E) > 0$. For example a system with n D0 and m D2p charges defines a projective module if $n + m\mathrm{Pf}\theta > 0$. In particular we may have negative n or m in some ranges allowed by the value of $\mathrm{Pf}\theta$. In that case we cannot represent such a module as a direct some of D0 and D2p modules and they have no solutions analogous to the solitons we have been considering so far. However these modules can be represented as a difference of modules in the sense of K-theory group. This suggests a physical interpretation of these modules as the ones describing a result of tachyon condensation of tachyons in D2p-$\overline{\mathrm{D0}}$ or $\overline{\mathrm{D2p}}$-D0 systems. For example consider the situation when $\mathrm{Pf}\theta > 0$ and a module has a noncommutative dimension $1 - \mathrm{Pf}\theta$. It is a real (as opposed to a virtual one) module when $\mathrm{Pf}\theta < 1$ or $\mathrm{Pf}(B/(2\pi)) > 1$ that is when the induced D0 charge due to the magnetic field on the D2p-brane can compensate the charge -1 of the anti D0-brane. Tachyon condensation processes for non-BPS branes and brane-antibrane systems on noncommutative tori were studied in [18, 19, 20].

5. Some open problems

For D0-D2 and D0-D4 systems that we discussed in detail we know the solitonic configuration describing a point-like D0 sitting on top of the higher-dimensional brane and we know that there exists on the same module an exact solution describing a BPS configuration in which D0 is smeared over the D2p brane. For D0-D2 system this is a constant curvature connection and for D0-D4 it is a noncommutative instanton. However we have not given an explicit construction of either of them. And the reason is that the direct sum representation of the module is not well suited for that purpose. Let us illustrate this point on the simplest example of D0-D2 system.

From general results on noncommutative tori [14] we know that any module over a two-dimensional torus (with irrational θ) admits a constant curvature connection. One can choose a gauge in which an arbitrary constant curvature connection takes the form $\nabla_j^{cc} + c_j \cdot \mathbf{1}$ where ∇_j^{cc} is some fiducial constant curvature connection and c_j are numbers. To find ∇_j^{cc} explicitly in the direct sum representation of our D0-D2 module we need to take a general connection in the form (2.9) and plug it into the equation $F_{12} = const \cdot \mathbf{1}$. In terms of components this is a system of 4 coupled equations that so far resisted our attempts to solve it directly. On the other hand there is another representation of the D0-D2 module in which it is very easy to construct such a connection. In this representation an element of E is represented by a single function $f(x) \in L_2(\mathbb{R}^1)$ and the torus generators act as

$$f(x)U_1 = f(x - \theta - 1)\,, \qquad f(x)U_2 = e^{2\pi i x} f(x)$$

and a constant curvature connection can be chosen as

$$\nabla_1^{cc} = \frac{2\pi i}{\theta + 1} x\,, \qquad \nabla_2^{cc} = \frac{\partial}{\partial x}\,.$$

We know that both representations of the D0-D2 module are isomorphic, the problem is to find an explicit isomorphism. One can write the following general ansatz for such an isomorphism $M : E_{0,1} \oplus E_{1,0} \to L_2(\mathbb{R}^1)$:

$$\begin{aligned} M &: (\psi(x), U_{\mathbf{n}}) \mapsto S\psi(x) + \chi(x) U_{\mathbf{n}}\,, \\ M^{-1} &: \phi(x) \mapsto (T\phi(x), \langle \tilde{\chi}\,, \phi \rangle_{T_\theta}) \end{aligned}$$

where S, T are some operators on $L_2(\mathbb{R}^1)$ and $\chi(x)$, $\tilde{\chi}(x)$ are some fixed functions. The condition that M^{-1} in the second line is indeed the inverse to M gives the following set of equations:

$$\begin{aligned} S^{\dagger}\tilde{\psi} &= 0\,, \qquad & T\psi &= 0\,, \\ TS &= 1\,, \qquad & ST &= 1 - \langle \tilde{\psi}, \psi \rangle_{T_{1/\theta}} \end{aligned} \tag{5.1}$$

from which it is clear that $\langle \tilde{\psi}, \psi \rangle_{T_{1/\theta}}$ is a projector that cuts out of $L_2(\mathbb{R}^1)$ a submodule isomorphic to $E_{0,1}$ and the complement submodule is a cyclic one spanned by the action of T_θ on a single vector $\psi(x)$. This construction is an analog of the partial isometry operators S and $S^\dagger$ used to construct solitons on the plane. So far we have been unable to find a suitable solution of these equations.

In [7] a consistent truncation of massive modes was proposed based on the symmetry properties of both the solitonic solution and the solution describing the final state of the tachyon condensation. It was shown that the massive modes can then be integrated out leading to an effective tachyon potential. For a similar analysis for the tachyon condensation on a torus one seems to need at least an explicit solution to (5.1). We leave this problem among others for a future investigation.

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